Received: *March 9, 2000,* Accepted: *April 10, 2000*

Modular invariant critical superstrings on four-dimensional Minkowski space $\times$ two-dimensional black hole

Shun'ya Mizoguchi

Institute of Particle and Nuclear Studies
High Energy Accelerator Research Organization (KEK)
Oho 1-1, Tsukuba, Ibaraki 305-0801, Japan
E-mail: shunya.mizoguchi@kek.jp

Abstract: Extending the seminal work of Bilal and Gervais, we construct a tachyon-free, modular invariant partition function for critical superstrings on four-dimensional Minkowski $\times$ two-dimensional black hole. This model may be thought of as an $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ version of Gepner models and corresponds to a conifold point on the moduli space of Calabi-Yau compactifications. We directly deal with $N = 2$, $c = 9$ unitary superconformal characters. Modular invariance is achieved by requiring the string to have a momentum along an extra noncompact direction, in agreement with the picture of singular CFTs advocated by Witten. The four-dimensional massless spectrum coincides with that of the tensionless strings, supporting the proposal made by Bershadsky et al. concerning a dual description of type-II strings on a conifold in terms of two intersecting NS5-branes. An interesting relation to $D = 6$, $N = 4$ gauged supergravity is also discussed.

Keywords: Conformal Field Models in String Theory, Supergravity Models.

JHEP04(2000)014

Contents

1. Introduction **1**

2. SL(2, ℝ)/ U(1) **Kazama-Suzuki model** **3**
- 2.1 Free-field realizations 3
- 2.2 Representations of classical SL(2, ℝ) 4
- 2.3 Interpretation — the SL(2, ℝ) WZW model 5

3. Modular invariant partition function **6**

4. Mass spectra **11**

5. Conclusion **12**

1. Introduction

One of the issues of string theory preventing phenomenological applications is the moduli that we get upon compactification. Since the critical dimension is $D = 10$ for (worldsheet) $N = 1$ superstrings, one has to compactify six of ten dimensions to get four-dimensional spacetime. Then there appear a number of massless scalars which characterize the size, the shape and other structures of the compactification space, although no so many (approximately) massless scalars are expected to be observed in reality. One must also break supersymmetries. It would therefore be interesting to explore any possibility to find a lower-dimensional string model with less supersymmetries which we could use as a starting point.

Recently, an interesting duality between strings on singular Calabi-Yau spaces and lower-dimensional non-gravitational (string) theories has been discussed [1]–[4]. In this paper, we will construct a modular invariant string partition function on four-dimensional Minkowski × two-dimensional black hole [5]. This model may be thought of as a non-compact SL(2, ℝ)/ U(1) version of Gepner models [6] and corresponds [7, 8] to a conifold point on the moduli space of Calabi-Yau compactifications of type-II string theories. We directly deal with the characters of $N = 2$, $c = 9$ unitary superconformal field theory realized as a Kazama-Suzuki model [9] based on SL(2, ℝ)/ U(1) [10] at level $k = 3$. Unlike the conventional Gepner models, which use a tensor product of minimal models (realized as a compact coset SU(2)/ U(1) [11]) as the internal CFT and is known [12, 13] to describe regular Calabi-Yau compactifications, the necessary central charge 9 is supplied by a single $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ conformal field theory. Thus we do not need to take a tensor product. If we regard the coset conformal field theory as a gauged $\mathrm{SL}(2,\mathbb{R})$ WZW model [14, 15], we end up with a six-dimensional string theory on four-dimensional Minkowski space $\times$ two-dimensional black hole.[1]

The main obstacle in constructing modular invariants using $N=2$, $c>3$ superconformal characters [17]–[19] was its bad modular behavior (see [20] for an earlier attempt); for example, the non-degenerate NS character is given by

$$\operatorname{Tr} q^{L_0} = q^{h+1/8}\,\frac{\vartheta_3(0|\tau)}{\eta^3(\tau)}\,, \tag{1.1}$$

whose monomial factor $q^{h+1/8}$ makes the modular behavior awful. To overcome this problem, we use the following two ideas: first, we consider (a countable set of) infinitely many primary fields so that the sum of their $q^{h+1/8}$ factors form a certain theta function. Which theta function to choose has to be examined carefully, and will be determined later. The modular property of the total partition function can be thus improved. Consistent CFTs with an infinite number of primary fields are known (e.g. $c=1$ CFTs [21, 22]) and not surprising. On the contrary, they are required for modular invariance [10].

There is still a problem even after the monomial of q is replaced by a theta function; the number of eta and theta functions do not balance between the denominator and the numerator. It is a problem because in the modular S transformation the $\sqrt{\tau}$ factors do not cancel. The situation is similar for a free scalar partition function, in which, however, the $|\tau|$ factor from the left and right eta functions is compensated by the modular transformation of the zeromode integral. Thus our second proposal is to assume that the "internal" $N=2$ superconformal system constructed from the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset has a degree of freedom of the center-of-mass motion along a certain direction, which has to be integrated over in the partition function. This assumption just agrees with the picture of CFTs on singular Calabi-Yau spaces advocated by Witten [23]. With these two ideas we construct a modular invariant, and it turns out that the integration over the continuous set of ensembles above is nothing but the "Liouville-momentum" integration along the cigar (see [24] for an earlier analysis of strings on two-dimensional black hole).

Another important question in constructing a model is how to restore the spacetime supersymmetry. In type-II string theories the key role was played by Jacobi's

[1] This is a *critical* string theory since the total central charge is 0 and the Liouville mode decouples. However, in the linear dilaton region which is far from the tip of the cigar, the coordinate field along the cigar looks like the Liouville field [5] which adjusts the total central to 0 if treated as a conformal field [16]. The idea of replacing the Liouville $\times$ $\mathrm{U}(1)$ system in non-critical string theories with two-dimensional euclidean black hole was proposed in [1].

abstruse identity. Some similar useful theta identities were found by Bilal and Gervais [25, 26] long time ago and were used to construct interesting non-critical superstring models. In fact, our model is closely related to their six-dimensional ($d = 5$) model, in particular contains the latter as a subsector, although the interpretation is somewhat different.

The remainder of this paper is organized as follows. In section 2, we briefly review the construction of $N = 2$ superconformal algebra based on the non-compact coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ and the geometrical meaning of the free fields used in the realization. In section 3, we use theta identities to construct a modular invariant partition function, clarify the $N = 2$ character content and summarize the features of our model. In section 4, we study the lowest mass spectra. Finally, we present our conclusions and suggestions for further work in section 5.

2. $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ Kazama-Suzuki model

2.1 Free-field realizations

Let us begin with a review of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ Kazama-Suzuki model [10]. We use the following free-field realization of the $\mathrm{SL}(2,\mathbb{R})$ current algebra:

$$\begin{aligned} J^3(z) &= i\sqrt{\frac{k}{2}}\partial\phi\,, \\ J^\pm(z) &= i\left(\sqrt{\frac{k}{2}}\partial\theta \pm i\sqrt{\frac{k-2}{2}}\partial\rho\right)\exp\left(\pm i\sqrt{\frac{2}{k}}(\theta-\phi)\right). \end{aligned} \tag{2.1}$$

The OPEs of the free scalars are $\rho(z)\rho(0) \sim \theta(z)\theta(0) \sim -\log z$ but $\phi(z)\phi(0) \sim +\log z$. This realization may be obtained by bosonizing [27] the β-γ system of the Wakimoto realization [28] followed by a redefinition of the scalars. The same realization was used in [29]. The energy-momentum tensor is

$$T_{\mathrm{SL}(2,\mathbb{R})}(z) = -\frac{1}{2}(\partial\rho)^2 + \frac{1}{\sqrt{2(k-2)}}\,\partial^2\rho - \frac{1}{2}(\partial\theta)^2 + \frac{1}{2}(\partial\phi)^2\,. \tag{2.2}$$

The central charge is $c_{\mathrm{SL}(2,\mathbb{R})} = 3k/(k-2)$.

The $\mathrm{SL}(2,\mathbb{R})$ parafermions are defined by removing the exponentials of ϕ from $J^\pm$:

$$\psi^\pm(z) = i\left(\sqrt{\frac{1}{2}}\,\partial\theta \pm i\sqrt{\frac{k-2}{2k}}\partial\rho\right)\exp\left(\pm i\sqrt{\frac{2}{k}}\,\theta\right), \tag{2.3}$$

where the parafermion fields $\psi^\pm$ are ψ_1 and $\psi_1^\dagger$ in the usual notation. (See also [30] for a realization of the $\mathrm{SL}(2,\mathbb{R})$ parafermion algebra).

The energy-momentum tensor of the parafermion theory is

$$T_{\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)}(z) = T_{\mathrm{SL}(2,\mathbb{R})}(z) - \left(+\frac{1}{2}(\partial\phi)^2\right). \tag{2.4}$$

The $N=2$ superconformal algebra can be obtained by adding back another free boson $\varphi(z)$ with the OPE $\varphi(z)\varphi(0) \sim -\log z$. The currents are given by

$$\begin{aligned}
T_{N=2}(z) &= T_{\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)}(z) - \frac{1}{2}(\partial\varphi)^2\,, \\
T_F^{\pm}(z) &= \sqrt{\frac{2k}{k-2}}\psi^{\pm}\exp\left(\pm i\sqrt{\frac{k-2}{k}}\varphi\right), \\
J_{N=2}(z) &= i\sqrt{\frac{k}{k-2}}\partial\varphi\,.
\end{aligned} \tag{2.5}$$

The central charge is $c_{N=2} = 3k/(k-2)$ again. k will be set to 3 later.

2.2 Representations of classical $\mathrm{SL}(2,\mathbb{R})$

A unitary $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset module is constructed by forbidding the J^3 excitation in a unitary highest (or rather lowest) weight module of the $\mathrm{SL}(2,\mathbb{R})$ current algebra. Thus the lowest L_0 level states form a unitary representation of (classical) $\mathrm{SL}(2,\mathbb{R})$. (See e.g. [31] for the representations of the classical $\mathrm{SL}(2,\mathbb{R})$ Lie algebra. See also [24].) The latter can be realized as the differential operators acting on functions on S^1:[2]

$$\begin{aligned}
H^3 &= \epsilon + i\frac{d}{dx}\,, \\
H^{\pm} &= ie^{\mp ix}\frac{d}{dx} + (\epsilon \pm l)e^{\mp ix}\,, \\
C &= \frac{1}{2}(H^+H^- + H^-H^+) - (H^3)^2\,.
\end{aligned} \tag{2.6}$$

$\{e^{-imx} \mid m \in \mathbb{Z}\}$ form a complete set and

$$\begin{aligned}
H^3 e^{-imx} &= (m+\epsilon)e^{-imx}\,, \\
H^{\pm} e^{-imx} &= (m+\epsilon \pm l)e^{-i(m\pm 1)x}\,, \\
C e^{-imx} &= -l(l-1)e^{-imx}\,.
\end{aligned} \tag{2.7}$$

Thus the representations of $\mathrm{SL}(2,\mathbb{R})$ are labeled by (l,ϵ). The corresponding equations in the $\mathrm{SL}(2,\mathbb{R})$ current algebra module are

$$\begin{aligned}
J_0^3|m+\epsilon\,\rangle &= (m+\epsilon)|m+\epsilon\,\rangle\,, \\
J_0^{\pm}|m+\epsilon\,\rangle &= (m+\epsilon\pm l)|m+\epsilon\pm 1\,\rangle\,, \\
\boldsymbol{J}|m+\epsilon\,\rangle &= -l(l-1)|m+\epsilon\,\rangle
\end{aligned} \tag{2.8}$$

with $\boldsymbol{J} = 1/2(J_0^+J_0^- + J_0^-J_0^+) - (J_0^3)^2$.

Unitary representations (l,ϵ) of (classical) $\mathrm{SL}(2,\mathbb{R})$ are known to be classified into the following four cases:[3]

[2] These equations correct the inconsistency in the sign convention in ref. [31].

[3] We do not need to consider the universal covering of $\mathrm{SL}(2,\mathbb{R})$ in our $c_{N=2} = 9$ case. Then ϵ takes discrete values.

JHEP04(2000)014

(i) *Principal unitary series*: $(+1/2+ip, \epsilon)$; $p \in \mathbb{R}$, $\epsilon \in \{0, 1/2\}$.

(ii) *Complementary series*: $(l, 0)$; $0 < l < 1$.

(iii+) *Discrete series* $\mathcal{D}_n^+$: $(n+\epsilon, \epsilon)$; $\epsilon \in \{0, 1/2\}$, $n \in \mathbb{Z}_{>0}$ if $\epsilon = 0$, $n \in \mathbb{Z}_{\geq 0}$ if $\epsilon = 1/2$.

(iii−) *Discrete series* $\mathcal{D}_n^-$: $(n-\epsilon, \epsilon)$; $n \in \mathbb{Z}_{\geq 0}$, $\epsilon \in \{0, 1/2\}$.

(iv) *Trivial representation.*

In the cases (i), (ii), the whole module consists of the states $|m+\epsilon\rangle, m \in \mathbb{Z}$. There are neither highest-weight states nor lowest-weight states. In the cases (iii±), the module spanned by $|m+\epsilon\rangle, m \in \mathbb{Z}$ turns reducible due to the appearance of null states. In this case the irreducible submodules $\mathcal{D}^{\pm}_{l\mp\epsilon}$ are unitary. $\mathcal{D}^+_{l-\epsilon}$ ($\mathcal{D}^-_{l+\epsilon}$) has a lowest- (highest-)weight state $|l\rangle$ ($|-l\rangle$). The quotient module divided by $\mathcal{D}^+_{l-\epsilon} \oplus \mathcal{D}^-_{l+\epsilon}$ is finite, and nonunitary if $l \neq 1/2, 1$. If $l = 1/2$ the quotient module is empty; if $l = 1$ the quotient module consists of a single state, and hence corresponds to the trivial representation. One can also construct unitary representations by starting from negative n, but they only give equivalent representations. ($\mathcal{D}_0^+$ is equivalent to $\mathcal{D}_1^+$ if $\epsilon = 0$.) This is manifest in the symmetry of the Casimir $l \leftrightarrow -l+1$. Finally, the trivial representation maps any element of the Lie algebra to 0.

The vertex operator of the $\mathrm{SL}(2,\mathbb{R})$ current algebra at level k corresponding to the state $|m+\epsilon\rangle$ in the (l, ϵ) representation is

$$|m+\epsilon\rangle \sim e^{+\sqrt{2/(k-2)}l\rho + i\sqrt{2/k}(m+\epsilon)(\theta-\phi)}\,. \tag{2.9}$$

It has

$$L_0^{\mathrm{SL}(2,\mathbb{R})} = -\frac{l^2-l}{k-2}, \qquad J_0^3 = m+\epsilon\,. \tag{2.10}$$

The corresponding $N=2$ vertex operator is

$$e^{+\sqrt{2/(k-2)}l\rho + i\sqrt{2/k}(m+\epsilon)\theta + i\left(2(m+\epsilon)/\sqrt{k(k-2)}\right)\varphi}\,. \tag{2.11}$$

It has

$$h = L_0^{N=2} = \frac{-(l^2-l)+(m+\epsilon)^2}{k-2}, \qquad Q = J_0^{N=2} = -\frac{2(m+\epsilon)}{k-2}\,. \tag{2.12}$$

2.3 Interpretation — the $\mathrm{SL}(2,\mathbb{R})$ WZW model

One of the nice features of the realization (2.1) is its clear geometrical meanings. To see this, let us write out the $\mathrm{SL}(2,\mathbb{R})$ WZW action

$$S_{\mathrm{WZW}} = \frac{k}{8\pi}\int d^2x \,\mathrm{Tr}\,\partial_\mu g\,\partial^\mu g^{-1} + k\Gamma(g)\,, \tag{2.13}$$

$$\Gamma(g) = \frac{1}{12\pi}\int d^3x\, \epsilon^{\overline{\mu\nu\rho}}\,\mathrm{Tr}\,\overline{g}^{-1}\partial_{\overline{\mu}}\overline{g}\;\overline{g}^{-1}\partial_{\overline{\nu}}\overline{g}\;\overline{g}^{-1}\partial_{\overline{\rho}}\overline{g} \tag{2.14}$$

using the parameterization

$$g = \begin{bmatrix} u+w & v+y \\ -v+y & u-w \end{bmatrix}, \tag{2.15}$$

$$u = \cosh \boldsymbol{\rho} \cos \boldsymbol{t}, \qquad v = \cosh \boldsymbol{\rho} \sin \boldsymbol{t},$$

$$w = \sinh \boldsymbol{\rho} \cos \tilde{\boldsymbol{\theta}}, \qquad y = \sinh \boldsymbol{\rho} \sin \tilde{\boldsymbol{\theta}}\,.$$

(The signature of the worldsheet is $\eta^{\mu\nu} = \mathrm{diag}[-1, +1]$). The result is

$$\begin{aligned} S_{\mathrm{WZW}} &= -\frac{k}{8\pi} \int d^2x \Big[\sqrt{-h} h^{\mu\nu} (\partial_\mu \boldsymbol{\rho} \partial_\nu \boldsymbol{\rho} - \cosh^2\!\boldsymbol{\rho} \partial_\mu \boldsymbol{t} \partial_\nu \boldsymbol{t} + \sinh^2\!\boldsymbol{\rho} \partial_\mu \tilde{\boldsymbol{\theta}} \partial_\nu \tilde{\boldsymbol{\theta}}) - \\ &\qquad - 2\epsilon^{\mu\nu} \sinh^2\!\boldsymbol{\rho} \partial_\mu \boldsymbol{t} \partial_\nu \tilde{\boldsymbol{\theta}} \Big] \\ &= \frac{k}{4\pi} \int d^2x \Big[-\partial_+ \boldsymbol{\rho} \partial_- \boldsymbol{\rho} + \partial_+ \boldsymbol{t} \partial_- \boldsymbol{t} + \sinh^2\!\boldsymbol{\rho} \partial_+ (\boldsymbol{t} - \tilde{\boldsymbol{\theta}}) \partial_- (\boldsymbol{t} + \tilde{\boldsymbol{\theta}}) \Big]\,. \end{aligned} \tag{2.16}$$

The last line is the expression on the flat worldsheet with $x^\pm = (x^1 \pm x^0)/\sqrt{2}$. $\mathrm{SL}(2, \mathbb{R})$ invariant metric is given by

$$ds^2 = d\boldsymbol{\rho}^2 - \cosh^2 \boldsymbol{\rho} d\boldsymbol{t}^2 + \sinh^2 \boldsymbol{\rho} d\tilde{\boldsymbol{\theta}}^2 \tag{2.17}$$

in this parameterization. We now dualize $\tilde{\boldsymbol{\theta}}$ by adding

$$-\frac{k}{4\pi} \int d^2x\, \epsilon^{\mu\nu}\, \partial_\mu \boldsymbol{\theta}\, \partial_\nu \tilde{\boldsymbol{\theta}} \tag{2.18}$$

to S_{WZW}. The dual action reads [32, 33]

$$S^{\mathrm{dual}}_{\mathrm{WZW}} = -\frac{k}{8\pi} \int d^2x \sqrt{-h} \left[(\partial_\mu \boldsymbol{\rho})^2 - (\partial_\mu (\boldsymbol{t} + \boldsymbol{\theta}))^2 + \tanh^2\!\boldsymbol{\rho} (\partial_\mu \boldsymbol{\theta})^2 \right] \tag{2.19}$$

with the dilaton field

$$\Phi = -\log \cosh \boldsymbol{\rho} + \mathrm{const}\,. \tag{2.20}$$

Thus the target space of the dual sigma model is $\mathbb{R}\times$ two-dimensional black hole. The cigar geometry may be obtained by simply dropping the "time" coordinate $\boldsymbol{t}+\boldsymbol{\theta}$. In the region $\boldsymbol{\rho} \to \infty$, $\boldsymbol{\rho}$, $\boldsymbol{\theta}$ and $\boldsymbol{t} + \boldsymbol{\theta}$ correspond to the free fields ρ, θ and ϕ, respectively. Clearly, ρ plays the role of the Liouville field.

3. Modular invariant partition function

Having reviewed the $\mathrm{SL}(2, \mathbb{R})/\,\mathrm{U}(1)$ coset construction, we will now construct a modular invariant partition function. In type-II string theories, the key equation was Jacobi's abstruse identity:

$$\vartheta_3^4(0|\tau) - \vartheta_4^4(0|\tau) - \vartheta_2^4(0|\tau) = 0\,. \tag{3.1}$$

Are there any such nice theta identities for us, too? In fact, we may find one in the works on non-critical string theory done by Bilal and Gervais [25, 26] long time ago:

$$\Lambda_1(\tau) \equiv \Theta_{1,1}(\tau,0)\left(\vartheta_3^2(0|\tau)+\vartheta_4^2(0|\tau)\right)-\Theta_{0,1}(\tau,0)\,\vartheta_2^2(0|\tau)=0\,, \tag{3.2}$$

where

$$\Theta_{m,1}(\tau,\nu)=\sum_{n\in\mathbb{Z}} q^{(n+m/2)^2} z^{2(n+m/2)} \quad (m=0,1)\,, \tag{3.3}$$

$$q=\exp(2\pi i\tau), \qquad z=\exp(2\pi i\nu)$$

are the level-1 SU(2) theta functions. We also use another identity:

$$\Lambda_2(\tau) \equiv \Theta_{0,1}(\tau,0)\left(\vartheta_3^2(0|\tau)-\vartheta_4^2(0|\tau)\right)-\Theta_{1,1}(\tau,0)\,\vartheta_2^2(0|\tau)=0\,, \tag{3.4}$$

which is nothing but the modular S transform of (3.2). The determinant of the coefficient matrix of $\Theta_{m,1}(\tau,0)$ in (3.2) and (3.4) consistently vanishes (for they are non-zero functions) due to (3.1). Their modular properties are

$$\Lambda_1(\tau+1)=i\Lambda_1(\tau)\,, \qquad \Lambda_2(\tau+1)=-\Lambda_2(\tau)\,, \tag{3.5}$$

and

$$\begin{aligned}\Lambda_1(\tau) &= \frac{\exp(3\pi i/4)}{\sqrt{2}\tau^{3/2}}\Bigg(-\Lambda_1\Big(\frac{-1}{\tau}\Big)+\Lambda_2\Big(\frac{-1}{\tau}\Big)\Bigg),\\ \Lambda_2(\tau) &= \frac{\exp(3\pi i/4)}{\sqrt{2}\tau^{3/2}}\Bigg(+\Lambda_1\Big(\frac{-1}{\tau}\Big)+\Lambda_2\Big(\frac{-1}{tau}\Big)\Bigg).\end{aligned} \tag{3.6}$$

Thus

$$\left|\frac{\Lambda_1(\tau)}{\eta^3(\tau)}\right|^2+\left|\frac{\Lambda_2(\tau)}{\eta^3(\tau)}\right|^2 \tag{3.7}$$

is modular invariant.

Having found a building block, we now construct a partition function. We set $k=3$ from now on. For the NS sector, we collect the unitary representations [34, 35]

$$h=\frac{Q^2+1}{4}+p^2, \qquad Q\in\mathbb{Z} \tag{3.8}$$

and integrate over $p\in\mathbb{R}$ with the same weight (figure 1). Except when Q is odd and $p=0$, they all have non-degenerate characters:

$$\mathrm{Tr}_{\mathrm{NS}}\, q^{L_0^{N=2}} z^{J_0^{N=2}} = \mathrm{ch}_{\mathrm{NS}}(h,Q) = q^{p^2+(Q^2+1)/4} z^Q f^{\mathrm{NS}}(\tau,z)\,, \tag{3.9}$$

where

$$f^{\mathrm{NS}}(\tau,z)\equiv\prod_{n=1}^{\infty}\frac{(1+zq^{n-1/2})(1+z^{-1}q^{n-1/2})}{(1-q^n)^2} \quad \left(=q^{1/8}\frac{\vartheta_3(\nu|\tau)}{(\eta(\tau))^3}\right). \tag{3.10}$$

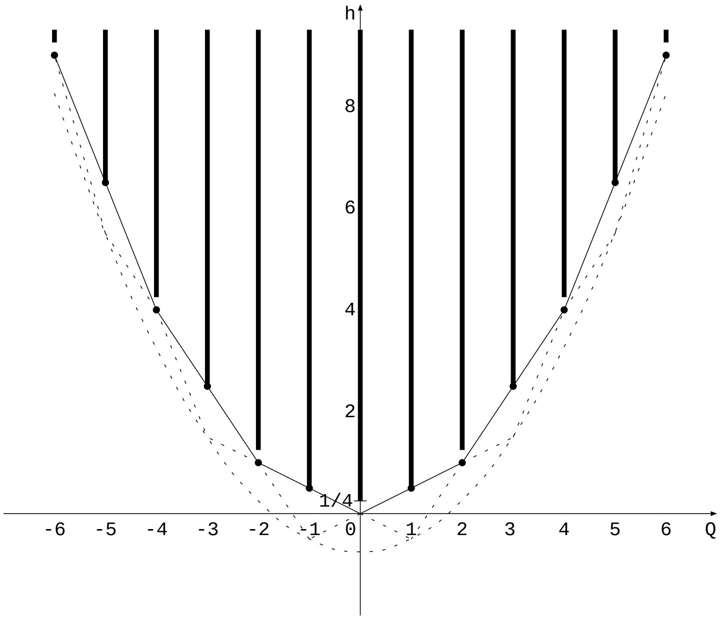

Figure 1: The NS sector. The thick lines and dots show the $N = 2$ unitary representations used as the internal CFT. The former correspond to the propagating modes along the cigar, while the latter are the bound states.

Note that this class of representations precisely correspond to the ones made out of the principal unitary series only.

When

$$Q = Q_m = -(2m+1), \qquad h = h_m = \frac{Q_m^2 + 1}{4} = m^2 + m + \frac{1}{2} \quad (m \in \mathbb{Z}) \tag{3.11}$$

(that is, $p = 0$), the representation reaches the boundary of the unitarity region, where its irreducible subspace becomes smaller. The non-degenerate character then decomposes [36] into a sum of two degenerate characters of A_3 type:

$$\mathrm{ch}_{\mathrm{NS}}(h_m, Q_m) = \mathrm{ch}_{\mathrm{A_3deg}}(h_m, Q_m; m) + \mathrm{ch}_{\mathrm{A_3deg}}\left(h_m + \left|m + \frac{1}{2}\right|, Q_m + \mathrm{sign}(Q_m); m\right), \tag{3.12}$$

where

$$\mathrm{ch}_{\mathrm{A_3\,deg}}(h, Q; r) = q^h z^Q \frac{f^{\mathrm{NS}}(\tau, z)}{1 + q^{|r+1/2|} z^{\mathrm{sign}(Q)}} \,. \tag{3.13}$$

The first term of (3.12) is the irreducible character of the representation (h_m, Q_m), while the second is that of the adjacent integer-Q degenerate representation on the same boundary line (figure 1). These representations are also summed over. As a result, the points (h_m, Q_m) may also be thought of as if they were generic (non-degenerate) ones. Such degenerate representations on the boundary lines of the unitarity region are made out of the discrete series of $\mathrm{SL}(2, \mathbb{R})$.

JHEP04(2000)014

For the R sector, we similarly consider the following set of representations:

$$h = \frac{Q^2}{4} + p^2, \qquad Q \in \mathbb{Z}\,. \tag{3.14}$$

In this case, unless Q is even and $p = 0$, the characters are given by the generic ones:

$$\mathrm{Tr}_{\mathrm{R}}\, q^{L_0^{N=2}} z^{J_0^{N=2}} = \mathrm{ch}_{\mathrm{R}}(h, Q) = q^{p^2 + (Q^2)/4} z^Q f^{\mathrm{R}}(\tau, z)\,, \tag{3.15}$$

where

$$f^{\mathrm{R}}(\tau, z) \equiv (z^{1/2} + z^{-1/2}) \prod_{n=1}^{\infty} \frac{(1 + zq^n)(1 + z^{-1}q^n)}{(1 - q^n)^2} \qquad \left(= \frac{\vartheta_2(\nu|\tau)}{(\eta(\tau))^3} \right). \tag{3.16}$$

We do not shift the U(1) charge by $\pm 1/2$ in the definition of the Ramond character; if the representation has two degenerate lowest L_0 states (which is the generic ($h \neq 3/8$) case), Q represents the *mean* value of U(1) charges of the two states. If $h = 3/8$, the lowest L_0 state is unique because the other becomes null. Then the U(1) charge of the lowest L_0 state is $Q - 1/2$ (P^+ module).

If Q is even and $p = 0$, the non-degenerate character again can be written as a sum of two irreducible degenerate characters of P_3^+ type:

$$\begin{aligned} \mathrm{ch}_{\mathrm{R}}(h_m, Q_m) = &\ \mathrm{ch}_{\mathrm{P}_3^+\,\mathrm{deg}}(h_m, Q_m; m) + \\ &+ \mathrm{ch}_{\mathrm{P}_3^+\,\mathrm{deg}}\left(h_m + |m|, Q_m + \mathrm{sign}\left(Q_m + \frac{1}{2}\right); m \right), \end{aligned} \tag{3.17}$$

where

$$Q = Q_m = -2m, \qquad h = h_m = \frac{Q_m^2}{4} = m^2 \quad (m \in \mathbb{Z}) \tag{3.18}$$

and

$$\mathrm{ch}_{\mathrm{P}_3^+\mathrm{deg}}(h, Q; r) = q^h z^Q \frac{f^{\mathrm{R}}(\tau, z)}{1 + q^{|r|} z^{\mathrm{sign}(Q+1/2)}} \tag{3.19}$$

(figure 2). Again, taking into account the extra degenerate representations (the second term of (3.17)), the points (h_m, Q_m) can be thought of as generic.

The two sets of representations (3.8) and (3.14), as well as the extra degenerate representations added to them, transform into each other by a spectral flow.

Which GSO projection should we take? We wish to construct a partition function in which the fermion theta in the four-dimensional Minkowski + ghost sector and the characters of the internal $N = 2$ sector are combined into the form like (3.7). Thus we need to have the following GSO projection: let F ($\overline{F}$) be the right (left) fermion number of the four-dimensional Minkowski + ghost sector, Q ($\overline{Q}$) the right (left) $N = 2$ U(1) charge and ϵ ($\bar{\epsilon}$) the parameter which distinguishes the parity of the U(1) charge of the right (left) $N = 2$ ground state (see eq. (2.11)). Then for the

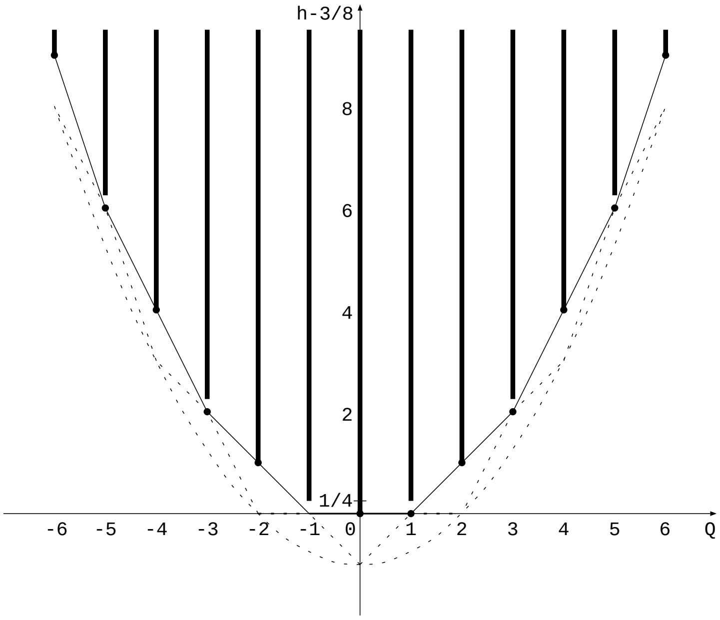

Figure 2: The R sector. Q represents the mean value of U(1) charges of two lowest L_0 states if $h \neq 3/8$. The U(1) charge of the lowest weight state is $Q \pm 1/2$ depending on the convention (which of $\mathrm{P}^{\pm}$ representations is considered). If $h = 3/8$, the lowest L_0 state is unique because the other becomes null. The (superficial) asymmetry at the bottom is due to the convention used here; we consider P^{+} representations so that the U(1) charge of the lowest weight state is $Q - 1/2$.

NS sector, we only keep the states with both $F + Q$ and $\overline{F} + \overline{Q}$ odd, *and* $2\epsilon + 2\overline{\epsilon}$ even. In other words, we keep odd fermion excited states if $\epsilon = \overline{\epsilon} = 0$, while we do even states if $\epsilon = \overline{\epsilon} = 1/2$. The two parameters m and ϵ labeling the U(1) charge are separately GSO projected. On the other hand, for the R sector, the states with the same $N = 2$ vacuum U(1) parity (i.e. $\epsilon = \overline{\epsilon}$) are similarly paired, but the left-right chirality may or may not be the same. If the chirality is the same, we get a IIB-like model, while if it is opposite, we get a IIA-like model. In addition, the left-right diagonal $p = \overline{p}$ are required for modular invariance for both sectors.

With this GSO projection, the total partition function reads

$$
\begin{aligned}
Z = \int \frac{d\tau d\overline{\tau}}{\operatorname{Im}\tau} (\operatorname{Im}\tau)^{-2} \left|\eta(\tau)\right|^{-4} (\operatorname{Im}\tau)^{-1/2} \left|\eta(\tau)\right|^{-2} \times \\
\times \left[\left| \frac{\Lambda_1(\tau)}{\eta^3(\tau)} \right|^2 + \left| \frac{\Lambda_2(\tau)}{\eta^3(\tau)} \right|^2 \right], \qquad (3.20)
\end{aligned}
$$

where the factor $(\operatorname{Im}\tau)^{-1/2}$ comes from the diagonal "Liouville-momentum" p integration, and $|\eta(\tau)|^{-2}$ from the transverse fermions. The transverse fermion theta has already been taken into account in the last factor and GSO projected with the $N = 2$ theta together. This is the main result of this paper.

We will now list some of the notable features of our partition function Z (3.20):

1. *It is modular invariant.* Modular invariance has been achieved by integrating the "Liouville momenta" p. This may be understood as a summation over the radial momenta on the cigar. Indeed, the principal unitary series is the only class of representations that corresponds to an $N=2$ vertex operator with *real* ρ momentum (apart from the imaginary background charge $-i/\sqrt{2}$). Consequently, spacetime, which was supposed to be four-dimensional, turns five-dimensional effectively; any "particle" in the four-dimensional world has a continuous spectrum. This agrees with the picture of singular CFTs advocated in ref. [23].

2. *It is unitary and tachyon free.*

3. *It is spacetime supersymmetric.* Z is zero in reality. Since the vanishing $\Lambda_{1,2}(\tau)$ are a consequence of the ordinary spectral flow, the spacetime supercharge must be given by the usual one using the bosonized ghost, the fermion-number current and the $N=2$, U(1) current. This is in contrast to the one in ref. [37] containing a contribution from the "longitudinal" boson.

4. *It has a graviton.* The tensor product of the NS transverse fermion excitations yields a graviton, a dilaton and an anti-symmetric two-form field. They survive the GSO projection. This is the most significant difference between Bilal-Gervais's model [25, 26] and ours; the graviton comes from the $F+Q-2\epsilon$ odd sector ($\Lambda_2(\tau)$), which is missing in the former. Those fields are massive in the sense that they have $L_0=\overline{L_0}=1/4$ even when $p=\overline{p}=0$.

5. *It contains bound states in the spectrum.* As we have shown in (3.12) and (3.17), the partition function has a contribution from the representations made out of the discrete series of $\mathrm{SL}(2,\mathbb{R})$. They do not have a momentum along the cigar and are regarded as the bound states [24].

4. Mass spectra

Let us now discuss more in detail the lightest mass spectra of our model. In the last section, we have seen that any four-dimensional particle exhibits a continuous mass spectrum, which is attributed to its momentum along the cigar. Therefore we consider the value of total L_0 of particles "at rest" along the cigar (i.e. $p=0$). This is equivalent to studying masses in five dimensions.

Since the transverse fermion theta and the $N=2$ fermion theta are GSO projected together and enter in the partition function symmetrically, SO(4) (acting on four real fermions) plays an analogous role to the transverse rotational group (or the little group for massive states) in six dimensions, although our model does not have six-dimensional Poincare invariance. The four-dimensional field content can be conveniently obtained by a dimensional reduction (since the $N = 2$ fermions carry no spacetime indices).

We first consider the states coming from $|\Lambda_1(\tau)|^2$. The lightest NS-NS fields are four scalars. They are of course common in both types of GSO projection in the R sector. On the other hand, the doubly-degenerate lowest L_0 states in the R sector cannot be a spinor of SO(4) because a pseudo-real spinor needs four components. Thus it can only be a non-chiral Majorana SO(2) spinor. Then the lightest R-R fields are a four-dimensional vector and two scalars in either projection. (The IIB- and IIA-like projections yield the same R-R fields here because either of the eigenspaces of the SO(4) chirality operator decompose into a direct sum of $+$ and $-$ SO(2) chirality eigenspaces). They are massless ($L_0 = \overline{L_0} = 0$). Including fermions, they form a four-dimensional $N = 2$, U(1) vector multiplet + a hypermultiplet [38]. They are the same field content as a single $N = 4$, U(1) vector multiplet has, although we have only eight spacetime supersymmetries from the standard supercharge construction. It is interesting that those fields are formally obtained by a dimensional reduction of a six-dimensional (2,0) tensor multiplet or a (1,1) vector multiplet on a flat background. They are the lightest fields of Bilal-Gervais's closed string model [26].

The lightest NS-NS bosons from $|\Lambda_2(\tau)|^2$ are a graviton, a dilaton and a 2-form (self-dual + anti-self-dual) as we have seen in the previous section. In the R sector, we have now twice as many states as those in $|\Lambda_1(\tau)|^2$ at the lowest level, and hence may regard them as components of a single SO(4) Weyl spinor. Then in the R-R sector, the IIB-like projection yields four anti-self-dual 2-forms and four scalars, while the IIA-like projection gives four vectors. They are the bosonic fields of a six-dimensional $N = 2$ ("$\mathcal{N} = 1$") graviton + a self-dual tensor + four anti-self-dual tensor multiplets (IIB-like), and a graviton + a self-dual tensor + four vector multiplets (IIA-like), respectively. Again, they combine into a (2,0) graviton + a tensor multiplets in the former case, and a single (1,1) graviton multiplet in the latter. The four-dimensional field contents are obtained by the dimensional reduction of those fields. They have $L_0 = \overline{L_0} = 1/4$.

5. Conclusion

In this paper we have constructed a modular invariant partition function of superstrings on four-dimensional Minkowski space $\times$ two-dimensional black hole using the $N = 2$, $c = 9$ superconformal characters. Our model may be thought of as a modular invariant extension of Bilal-Gervais's $d = 5$ non-critical string model and describes type-II strings on a conifold. It is unitary, tachyon free and has a continuous spectrum in the four-dimensional sense.

In the $\alpha' \to 0$ limit the cigar becomes very thin and can be replaced by a thin cylinder $\mathbb{R} \times S^1$ since one would need very high energy to see the effect of the sharp tip. In this case one is left with a four-dimensional $N = 2$ non-gravitational theory with a vector multiplet and a hypermultiplet. This will give an example of holography proposed in [1, 2], and the $\mathrm{SL}(2, \mathbb{R})$ coset in our model will provide a regularization of the strong coupling singularity of the linear-dilaton vacuum.

It is also interesting that the four-dimensional massless spectrum (with $p = 0$) of our model coincides with that of the tensionless strings which arise on the four-dimensional intersection of two M5-branes [39]. To understand its implication we recall that the (0,4) tensionless strings arise in type-IIB theory on K3 when K3 gets an ADE singularity [23], while IIB on such a singular K3 is known [8] to be T-dual to a system of type-IIA 5-branes [40]. Thus, in view of this, the coincidence of the spectra supports the proposal in [41] that type-II strings on a Calabi-Yau three-fold with a conifold singularity have a dual description in terms of two intersecting NS5-branes.

On the other hand, $\alpha' \to \infty$ means that the area near the tip of the cigar is zoomed in on, and the whole target space looks like a six-dimensional Minkowski space. In this case all the towers of particles affect the low-energy physics, of which we cannot expect any local quantum field theory descriptions.

In the middle region of α' in between, the lowest level fields do not completely decouple but interact with some other light fields. In the previous section, we have seen that the first two lightest fields in the IIA-like GSO projection are the dimensional reduction of a six-dimensional $(1, 1)$ vector multiplet and a graviton multiplet. Remarkably, they are the field content of $D = 6$, $N = 4$ $(= (1, 1))$ gauged supergravity [42, 43]! Although the graviton multiplet has $L_0 = \overline{L_0} = 1/4$, perhaps this is linked to the well-known subtlety in defining masslessness of a particle in a curved background (see e.g. [44]). Indeed, $L_0 + \overline{L_0}$ corresponds to the Klein-Gordon operator in a flat space, but the correspondence becomes less clear in the Minkowski $\times$ cigar geometry. If the massless point is shifted by $1/4$, the graviton becomes massless but the lightest fields from $|\Lambda_1(\tau)|^2$ then have negative mass square. They can, however, still remain stable if they are above the Breitenlohner-Freedman bound [45]. It would be interesting to explore $D = 6$, $N = 4$ gauged supergravity on this background. The supergravity interpretation of the IIB-like projection model remains an open question.

Finally, some generalization of our construction to other singular Calabi-Yau spaces can be done when the corresponding analogue of Jacobi's abstruse identity is known. For example, an interesting theta identity was found in [46], which seems to be related to a Calabi-Yau four-fold with a singularity. Very recently, new theta identities corresponding to Calabi-Yau n-folds with an ADE singularity have been systematically obtained by Eguchi and Sugawara [47].

JHEP04(2000)014

Acknowledgments

I wish to thank T. Eguchi, A. Fujii, T. Kawai, Y. Satoh, Y. Yamada and S.-K. Yang for valuable discussions. I also wish to thank K. Fujikawa, Y. Matsuo, Y. Sugawara and all participants of the seminar given at Department of Physics, University of Tokyo in Nov. 1999 for questions and comments, which were useful for completing the final version. I am also grateful for the hospitality of Department of Mathematics, Kobe University where a part of this work was done.

References

[1] O. Aharony, M. Berkooz, D. Kutasov and N. Seiberg, *Linear dilatons, NS5-branes and holography*, *J. High Energy Phys.* **10** (1998) 004 [hep-th/9808149].

[2] A. Giveon, D. Kutasov and O. Pelc, *Holography for non-critical superstrings*, *J. High Energy Phys.* **10** (1999) 035 [hep-th/9907178].

[3] A. Giveon and D. Kutasov, *Little string theory in a double scaling limit*, *J. High Energy Phys.* **10** (1999) 034 [hep-th/9909110]; *Comments on double scaled little string theory*, *J. High Energy Phys.* **01** (2000) 023 [hep-th/9911039].

[4] O. Pelc, *Holography, singularities on orbifolds and 4D* $N = 2$ *SQCD*, *J. High Energy Phys.* **03** (2000) 012 [hep-th/0001054].

[5] E. Witten, *On string theory and black holes*, *Phys. Rev.* **D 44** (1991) 314.

[6] D. Gepner, *Space-time supersymmetry in compactified string theory and superconformal models*, *Nucl. Phys.* **B 296** (1988) 757;
Exactly solvable string compactifications on manifolds of $\mathrm{SU}(N)$ *holonomy*, *Phys. Lett.* **B 199** (1987) 380.

[7] D. Ghoshal and C. Vafa, $c = 1$ *string as the topological theory of the conifold*, *Nucl. Phys.* **B 453** (1995) 121 [hep-th/9506122].

[8] H. Ooguri and C. Vafa, *Two-dimensional black hole and singularities of CY manifolds*, *Nucl. Phys.* **B 463** (1996) 55 [hep-th/9511164].

[9] Y. Kazama and H. Suzuki, *New n=2 superconformal field theories and superstring compactification*, *Nucl. Phys.* **B 321** (1989) 232.

[10] L.J. Dixon, M.E. Peskin and J. Lykken, *N=2 superconformal symmetry and* $\mathrm{SO}(2,1)$ *current algebra*, *Nucl. Phys.* **B 325** (1989) 329.

[11] Z. an Qiu, *Nonlocal current algebra and* $N = 2$ *superconformal field theory in two-dimensions*, *Phys. Lett.* **B 188** (1987) 207;
A.B. Zamolodchikov and V.A. Fateev, *Disorder fields in two-dimensional conformal quantum field theory and* $N = 2$ *extended supersymmetry*, *Sov. Phys. JETP* **63** (1986) 913.

[12] E.J. Martinec, *Algebraic geometry and effective lagrangians*, *Phys. Lett.* **B 217** (1989) 431.

[13] C. Vafa and N. Warner, *Catastrophes and the classification of conformal theories*, *Phys. Lett.* **B 218** (1989) 51.

[14] H.J. Schnitzer, *A path integral construction of superconformal field theories from a gauged supersymmetric Wess-Zumino-Witten action*, *Nucl. Phys.* **B 324** (1989) 412.

[15] E. Witten, *The N matrix model and gauged WZW models*, *Nucl. Phys.* **B 371** (1992) 191.

[16] J. Distler and H. Kawai, *Conformal field theory and 2D quantum gravity or who's afraid of Joseph Liouville?*, *Nucl. Phys.* **B 321** (1989) 509.

[17] V.K. Dobrev, *Characters of the unitarizable highest weight modules over the $N=2$ superconformal algebras*, *Phys. Lett.* **B 186** (1987) 43.

[18] Y. Matsuo, *Character formula of $c<1$ unitary representation of $N=2$ superconformal algebra*, *Prog. Theor. Phys.* **77** (1987) 793.

[19] E. Kiritsis, *Character formulae and the structure of the representations of the $N=1$, $N=2$ superconformal algebras*, *Int. J. Mod. Phys.* **A 3** (1988) 1871.

[20] K. Huitu, *Modular invariants of $N=2$ supersymmetric* SU(1,1) *models*, *Phys. Lett.* **B 313** (1993) 75 [hep-th/9206008].

[21] S.-K. Yang, *Modular invariant partition function of the ashkin-teller model on the critical line and n=2 superconformal invariance*, *Nucl. Phys.* **B 285** (1987) 183.

[22] P. Ginsparg, *Curiosities at $c=1$*, *Nucl. Phys.* **B 295** (1988) 153.

[23] E. Witten, *Some comments on string dynamics*, In Los Angeles 1995, *Future perspectives in string theory*, p. 501 [hep-th/9507121].

[24] R. Dijkgraaf, H. Verlinde and E. Verlinde, *String propagation in a black hole geometry*, *Nucl. Phys.* **B 371** (1992) 269.

[25] A. Bilal and J.L. Gervais, *New critical dimensions for string theories*, *Nucl. Phys.* **B 284** (1987) 397.

[26] A. Bilal and J.L. Gervais, *Modular invariance for closed strings at the new critical dimensions*, *Phys. Lett.* **B 187** (1987) 39.

[27] D. Friedan, E. Martinec and S. Shenker, *Conformal invariance, supersymmetry and string theory*, *Nucl. Phys.* **B 271** (1986) 93.

[28] M. Wakimoto, *Fock representations of the affine Lie algebra $A1_1$*, *Comm. Math. Phys.* **104** (1986) 605.

JHEP04(2000)014

[29] Y. Satoh, *Ghost-free and modular invariant spectra of a string in* $\mathrm{SL}(2,\mathbb{R})$ *and three-dimensional black hole geometry*, *Nucl. Phys.* **B 513** (1998) 213 [hep-th/9705208].

[30] P.A. Griffin and O.F. Hernandez, *Feigin-fuchs derivation of* $\mathrm{SU}(1,1)$ *parafermion characters*, *Nucl. Phys.* **B 356** (1991) 287.

[31] N.Ja. Vilenkin and A.U. Klimyk, *Representation of Lie groups and special functions*, vol. 1. Kluwer Academic Publishers 1991.

[32] T.H. Buscher, *A symmetry of the string background field equations*, *Phys. Lett.* **B 194** (1987) 59;
Path integral derivation of quantum duality in nonlinear sigma models, *Phys. Lett.* **B 201** (1988) 466.

[33] G.T. Horowitz and D.L. Welch, *Exact three-dimensional black holes in string theory*, *Phys. Rev. Lett.* **71** (1993) 328 [hep-th/9302126].

[34] P.D. Vecchia, J.L. Petersen and H.B. Zheng, $N=2$ *extended superconformal theories in two-dimensions*, *Phys. Lett.* **B 162** (1985) 327;
S. Nam, *The kac formula for the* $N=1$ *and the* $N=2$ *superconformal algebras*, *Phys. Lett.* **B 172** (1986) 323.

[35] W. Boucher, D. Friedan and A. Kent, *Determinant formulae and unitarity for the* $N=2$ *superconformal algebras in two-dimensions or exact results on string compactification*, *Phys. Lett.* **B 172** (1986) 316.

[36] T. Eguchi and A. Taormina, *On the unitary representations of* $N=2$ *and* $N=4$ *superconformal algebras*, *Phys. Lett.* **B 210** (1988) 125.

[37] D. Kutasov and N. Seiberg, *Noncritical superstrings*, *Phys. Lett.* **B 251** (1990) 67.

[38] A. Strominger, Massless black holes and conifolds in string theory, *Nucl. Phys.* **B 451** (1995) 96 [hep-th/9504090].

[39] A. Hanany and I.R. Klebanov, *On tensionless strings in* $3+1$ *dimensions*, *Nucl. Phys.* **B 482** (1996) 105 [hep-th/9606136].

[40] C.G. Callan Jr., J.A. Harvey and A. Strominger, *Supersymmetric string solitons*, hep-th/9112030.

[41] M. Bershadsky, C. Vafa and V. Sadov, *D-strings on D-manifolds*, *Nucl. Phys.* **B 463** (1996) 398 [hep-th/9510225].

[42] F. Giani, M. Pernici and P. van Nieuwenhuizen, *Gauged* $N=4$ $D=6$ *supergravity*, *Phys. Rev.* **D 30** (1984) 1680.

[43] L.J. Romans, *The* F_4 *gauged supergravity in six-dimensions*, *Nucl. Phys.* **B 269** (1986) 691.

[44] P. van Nieuwenhuizen, *The complete mass spectrum of $D = 11$ supergravity compactified on S_4 and a general mass formula for arbitrary cosets M_4*, *Class. and Quant. Grav.* **2** (1985) 1.

[45] P. Breitenlohner and D.Z. Freedman, *Positive energy in anti-de Sitter backgrounds and gauged extended supergravity*, *Phys. Lett.* **B 115** (1982) 197;
Stability in gauged extended supergravity, *Ann. Phys. (NY)* **144** (1982) 249.

[46] D. Kutasov, *Some properties of (non)critical strings*, `hep-th/9110041`.

[47] T. Eguchi and Y. Sugawara, *Modular invariance in superstring on Calabi-Yau N-fold with ADE singularity*, `hep-th/0002100`.